

Clifford Parallelisms and External Planes to the Klein quadric

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Dedicated to the memory of Günter Pickert

Abstract

For any three-dimensional projective space $\mathbb{P}(V)$, where V is a vector space over a field F of arbitrary characteristic, we establish a one-one correspondence between the Clifford parallelisms of $\mathbb{P}(V)$ and those planes of $\mathbb{P}(V \wedge V)$ that are external to the Klein quadric representing the lines of $\mathbb{P}(V)$. We also give two characterisations of a Clifford parallelism of $\mathbb{P}(V)$, both of which avoid the ambient space of the Klein quadric.

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1 Introduction

A theory of *Clifford parallelisms* on the line set of a projective space over a field F was developed already several decades ago; see [30, pp. 112–115] and [32, § 14] for a detailed survey. This theory is based upon *projective double spaces* and their algebraic description via left and right multiplication in two classes of F -algebras: (A) *quaternion skew fields* with centre F and (B) *purely inseparable field extensions of F* satisfying some extra property. More precisely, this theory is built up from *two* parallelisms, hence the name “double space”. These parallelisms are identical in case (B) and distinct otherwise. It is common to distinguish between them by adding the appropriate attribute “left” or “right”.

In the present paper we aim at giving characterisations of a *single* Clifford parallelism. We confine ourselves to the three-dimensional case. First, we collect the necessary background information in some detail, as it is widespread over the literature. In Section 4, we consider a three-dimensional projective space $\mathbb{P}(V)$ on a vector space V over a field F (of arbitrary characteristic) and the Klein quadric

\mathcal{Q} representing the lines of $\mathbb{P}(V)$. Each *external plane to the Klein quadric*, i. e. a plane in the ambient space of \mathcal{Q} that has no point in common with \mathcal{Q} , is shown to give rise to a parallelism of $\mathbb{P}(V)$. Polarising such a plane with respect to the Klein quadric yields a second plane, which is also external to \mathcal{Q} , and hence a second parallelism. We establish in Theorem 4.8 that these parallelisms turn $\mathbb{P}(V)$ into a projective double space, which therefore can be described algebraically in the way we sketched above. The main result in Section 5 is Theorem 5.1, where we reverse the previous construction. What this all amounts to is a *one-one correspondence between Clifford parallelisms and planes that are external to the Klein quadric*. Finally, in Section 6 we present two characterisations of a Clifford parallelism. Both of them avoid the ambient space of the Klein quadric, even though our proofs rely on the aforementioned correspondence. Theorem 6.2 characterises a parallelism as being Clifford via the property that any two distinct parallel classes are contained in a *geometric hyperplane* of the Grassmann space formed by the lines of $\mathbb{P}(V)$. A second characterisation is given in Theorem 6.3; it makes use of a presumably new criterion, namely what we call the *condition of crossed pencils*.

When dealing with Clifford parallelisms in general, one must not disregard the rich literature about the classical case over the real numbers. The recent articles [2] and [11] provide a detailed survey; further sources can be found in [5, p. 10] and [12]. Some of the classical results remain true in a more general setting, even though not in all cases. For example, a quadric without real points will carry two reguli after its complexification. Any such regulus can be used to characterise a Clifford parallelism in the real case. By [7], this result still applies (with some subtle modifications) in case (A) but, according to [25], it fails in case (B). A crucial question is therefore to find classical results that can be generalised without limitation. Many of our findings are of this kind, and we shall give references to related work over the real numbers in the running text.

Finally, for the sake of completeness, let us mention that higher-dimensional analogues of Clifford parallelisms can be found in [17] and [47].

2 Preliminaries

Let U be a vector space over a field¹ F . It will be convenient to let the *projective space* $\mathbb{P}(U)$ be the set of all subspaces of U with *incidence* being symmetrised inclusion. We adopt the usual geometric terms: If $Z \subset U$ is a subspace of U with vector dimension $k + 1$ then its *projective dimension* is k . *Points*, *lines*, *planes*, and *solids* are the subspaces of U with vector dimension one, two, three, and four,

¹We assume the multiplication in a field to be commutative, in a skew field it may be commutative or not.

respectively. For any subspace $Z \subset U$ we let $\mathcal{P}(Z)$ and $\mathcal{L}(Z)$ be the set of all points and lines, respectively, that are contained in Z . If Y and Z are subspaces such that $Y \subset Z \subset U$ then $\mathcal{L}(Y, Z) := \{M \in \mathcal{L}(Z) \mid Y \subset M\}$. In particular, if $p \subset U$ is a point then $\mathcal{L}(p, U)$ is the *star of lines* with centre p . If, furthermore, $Z \subset U$ is a plane incident with p then $\mathcal{L}(p, Z)$ is a *pencil of lines*. We now could formalise the given projective space in terms of points and lines, but refrain from doing so.

For the rest of the article, it will be assumed that V is a vector space over F with vector dimension four, and we shall be concerned with the projective space $\mathbb{P}(V)$.

The exterior square $V \wedge V$ has vector dimension six and gives rise to the projective space $\mathbb{P}(V \wedge V)$. (All the multilinear algebra we need can be found in standard textbooks, like [34, Sect. 10.4] or [33, Sect. 6.8] to mention but a few.) Upon choosing any basis e_0, e_1, e_2, e_3 of V , the six bivectors $e_{\sigma\tau} = e_\sigma \wedge e_\tau$, $0 \leq \sigma < \tau \leq 3$ constitute a basis of $V \wedge V$. Writing vectors in the form $u = \sum_{\sigma=0}^3 u_\sigma e_\sigma$, $v = \sum_{\tau=0}^3 v_\tau e_\tau$ with $u_\sigma, v_\tau \in F$ gives

$$u \wedge v = \sum_{\sigma < \tau} (u_\sigma v_\tau - u_\tau v_\sigma) e_{\sigma\tau}. \quad (2.1)$$

The following results can be found, among others, in [6, Sect. 11.4], [27, Sect. 15.4], [41, Ch. 34], and [43, Ch. xv]. The *Plücker embedding*

$$\gamma: \mathcal{L}(V) \rightarrow \mathcal{P}(V \wedge V): M \mapsto F(u \wedge v) \quad (2.2)$$

assigns to each line $M \in \mathcal{L}(V)$ the point $F(u \wedge v)$, where $u, v \in M$ are arbitrary linearly independent vectors. If u and v are expressed as above then the six coordinates appearing on the right hand side of (2.1) are the *Plücker coordinates* of the line M . The image $\mathcal{L}(V)^\gamma =: \mathcal{Q}$ is the well known *Klein quadric* representing the lines of $\mathbb{P}(V)$. It is given, in terms of coordinates $x_{\sigma\tau} \in F$, by the quadratic form

$$\omega: V \wedge V \rightarrow F: \sum_{\sigma < \tau} x_{\sigma\tau} e_{\sigma\tau} \mapsto x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}. \quad (2.3)$$

Polarisation of ω gives the non-degenerate symmetric bilinear form

$$\langle \cdot, \cdot \rangle: (V \wedge V)^2 \rightarrow F: (x, y) \mapsto (x + y)^\omega - x^\omega - y^\omega, \quad (2.4)$$

whose explicit expression in terms of coordinates is immediate from (2.3). Using the exterior algebra $\wedge V$, the form $\langle \cdot, \cdot \rangle$ can be characterised via

$$x \wedge y = \langle x, y \rangle (e_0 \wedge e_1 \wedge e_2 \wedge e_3) \text{ for all } x, y \in V \wedge V.$$

The bilinear form $\langle \cdot, \cdot \rangle$ defines the *polarity of the Klein quadric*, which sends any subspace $X \subset V \wedge V$ to X^\perp . One crucial property of \perp is as follows: Lines

$M, N \in \mathcal{L}(V)$ have a point in common precisely when M^γ and N^γ are *conjugate points* with respect to \perp , i. e., $N^\gamma \subset M^{\gamma\perp}$ (or vice versa).

A *linear complex of lines* of $\mathbb{P}(V)$ is a set $\mathcal{H} \subset \mathcal{L}(V)$ whose image \mathcal{H}^γ is a *hyperplane section* of \mathcal{Q} , i. e., $\mathcal{H}^\gamma = \mathcal{P}(W) \cap \mathcal{Q}$ for some hyperplane $W \subset V \wedge V$. The complex is called *special* if W is tangent to the Klein quadric, and *general* otherwise. A subset of $\mathcal{L}(V)$ is a general linear complex if, and only if, it is the set of null lines of a null polarity. All lines that meet or are equal to a fixed line $A \in \mathcal{L}(V)$ constitute a special linear complex with *axis* A ; all special linear complexes of $\mathbb{P}(V)$ are of this form. A *linear congruence of lines* of $\mathbb{P}(V)$ corresponds—via γ —to the section of \mathcal{Q} by a solid, say T . Such a congruence is said to be *elliptic* if its image under γ is an elliptic (or: *ovoidal* [42, 2.1.4]) quadric of $\mathbb{P}(T)$. This will be the case precisely when the line T^\perp has no point in common with \mathcal{Q} . The elliptic linear congruences are precisely the *regular spreads*. There are three more types of linear congruences, but they will not be needed here. The Klein quadric has two systems of generating planes. For any plane G of the first (resp. second) system there is a unique point p (resp. plane Z) in $\mathbb{P}(V)$ such that $\mathcal{L}(p, V)^\gamma = \mathcal{P}(G)$ (resp. $\mathcal{L}(Z)^\gamma = \mathcal{P}(G)$). Finally, we recall that a set $\mathcal{R} \subset \mathcal{L}(V)$ is a *regulus* if, and only if, there is a plane E of $\mathbb{P}(V \wedge V)$ such that $\mathcal{R}^\gamma = \mathcal{P}(E) \cap \mathcal{Q}$ is a non-degenerate conic. For a quick overview and a detailed description of these and other linear sections of the Klein quadric we refer to the table in [27, pp. 29–31], even though over an infinite field some modifications may apply.

The *Grassmann space* $\mathbb{G}_1(V) := (\mathcal{L}(V), \Pi(V))$ is that partial linear space whose “point set” is the set $\mathcal{L}(V)$ of lines of $\mathbb{P}(V)$ and whose “line set” is the set $\Pi(V)$ of all pencils of lines in $\mathbb{P}(V)$. See, for example, [38, p. 71]. For the sake of readability we henceforth shall address the “points” and “lines” of $\mathbb{G}_1(V)$ by their original names. A *geometric hyperplane* \mathcal{H} of the Grassmann space $\mathbb{G}_1(V)$ is a proper subset of $\mathcal{L}(V)$ such that each pencil of lines either contains a single element of \mathcal{H} or is entirely contained in \mathcal{H} . The geometric hyperplanes of $\mathbb{G}_1(V)$, which are also called *primes*, are precisely the linear complexes of lines of $\mathbb{P}(V)$. Many proofs were given for this result (and its generalisation to other Grassmann spaces) [4], [13], [16] (finite ground field, see also [27, Thm. 15.2.14]), [20, p. 179] (rephrased in [26, Prop. 3]), and [44].

3 Parallelisms

Let $(\mathcal{P}, \mathcal{L})$ be a projective space with point set \mathcal{P} and line set \mathcal{L} . An equivalence relation $\parallel \subset \mathcal{L} \times \mathcal{L}$ is a *parallelism* if each point $p \in \mathcal{P}$ is incident with precisely one line from each equivalence class. The equivalence classes of \parallel are also called *parallel classes*. For further information see [30] and [31].

One class of parallelisms is based on the following notions: Let H be an alge-

bra over a field F such that one of the subsequent conditions² is satisfied:

(A) H is a quaternion skew field with centre F .

(B) H is an extension field of F with degree $[H : F] = 4$ and such that $h^2 \in F$ for all $h \in H$.

In both cases, H is an infinite *quadratic* (or: *kinematic*) F -algebra, i. e., $h^2 \in F + Fh$ for all $h \in H$.

Remark 3.1. Let us briefly recall a few facts about the F -algebras appearing in conditions (A) and (B):

Ad (A): Any *quaternion skew field* H with centre F arises as follows [46, pp.46–48]: We start with a separable quadratic field extension K/F and denote by $\bar{\cdot} : K \rightarrow K : z \mapsto \bar{z}$ the only non-trivial automorphism of K that fixes F elementwise. Also, we assume that there is an element $b \in F$ satisfying $b \neq z\bar{z}$ for all $z \in K$. Then $(K/F, b)$ is the subring of the matrix ring $K^{2 \times 2}$ comprising all matrices of the form

$$\begin{pmatrix} z & w \\ b\bar{w} & \bar{z} \end{pmatrix} \text{ with } z, w \in K \text{ arbitrary.} \quad (3.1)$$

We identify $z \in K$ with the matrix $\text{diag}(z, \bar{z}) \in (K/F, b)$ and, finally, we adopt the notation $H := (K/F, b)$.

The *conjugation* $\bar{\cdot} : H \rightarrow H$ is that antiautomorphism of H which takes a quaternion as in (3.1) to

$$\begin{pmatrix} \bar{z} & -w \\ -b\bar{w} & z \end{pmatrix}, \quad (3.2)$$

thereby extending the mapping $\bar{\cdot}$ from above, whence the notation is unambiguous. The multiplicative *norm function* $H \rightarrow F : h \mapsto h\bar{h} = \bar{h}h$ is a quadratic form on the F -vector space H . The F -linear form $H \rightarrow F : h \mapsto h + \bar{h}$ is the *trace function*. Any $h \in H$ satisfies the quadratic equation $h^2 - (h + \bar{h})h + h\bar{h} = 0$ with coefficients in F . Upon choosing any $i \in K \setminus F$ the matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & \bar{i} \end{pmatrix}, \quad j := \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}, \quad k := \begin{pmatrix} 0 & i \\ b\bar{i} & 0 \end{pmatrix} \quad (3.3)$$

constitute a basis of H over F . It is conventional to choose i as follows:

(A1) $\text{Char } F \neq 2$: We may assume that $i^2 - a = 0$ for some $a \in F \setminus \{0\}$, whence $\bar{i} = -i$, $\bar{j} = -j$, and $\bar{k} = -k$. This gives the formulas

$$\begin{aligned} i^2 &= a, & j^2 &= b, & k^2 &= -ab, \\ ij &= -ji = k, & jk &= -kj = -bi, & ki &= -ik = -aj. \end{aligned} \quad (3.4)$$

²Below we identify F with $F \cdot 1_H \subset H$ via $f \equiv f \cdot 1_H$ for all $f \in F$.

(A2) Char $F = 2$: We may assume that $i^2 + i + a = 0$ for some $a \in F \setminus \{0\}$, whence $\bar{i} = i + 1$, $\bar{j} = j$, and $\bar{k} = k$. Now we obtain

$$\begin{aligned} i^2 &= i + a, & j^2 &= b, & k^2 &= ab, \\ ij &= k, & jk &= b + bi, & ki &= aj, \\ ji &= j + k, & kj &= bi, & ik &= aj + k. \end{aligned} \tag{3.5}$$

We refer to [40, p. 169] for a different basis of H over F . It has the advantage to be applicable in any characteristic for quaternion skew fields, but using it would not allow us to incorporate case (B) in the way we do below. See also [36] for a characterisation of arbitrary quaternion skew fields. An analogous characterisation for real quaternions may be found in [18, p. 43].

Ad (B): Here Char $F = 2$, since there exists an $h \in H \setminus F$ for which $(1 + h)^2 = 1 + 2h + h^2 \in F$ implies $2h \in F$, and so $2 = 0$. The field H is a purely inseparable extension of F , and it fits formally into the description from (A1) if we proceed as follows: First, we select arbitrary elements $i, j \in H$ such that $1, i, j$ are linearly independent over F . Next, we let $k := ij$, $a := i^2 \in F \setminus \{0\}$, $b := j^2 \in F \setminus \{0\}$, and we regard the identity mapping as being the *conjugation* $\bar{\cdot} : H \rightarrow H$. Then, taking into account that minus signs can be ignored due to Char $F = 2$, the multiplication in H is given by the formulas in (3.4). (One may also carry over formulas (3.1) (3.2), and (3.3) by letting $K := F(i) \subset H$ and $\bar{z} := z$ for all $z \in K$.) The *norm* and *trace* of $h \in H$ are defined as for quaternions. So the norm of $h \in H$ is h^2 and its trace is $h + h = 0$. The polar form of the quadratic norm form $H \rightarrow F$ is the zero bilinear form.

We now consider the (three-dimensional) projective space $\mathbb{P}(H)$ on the F -vector space H . Our next definition, which follows [30, pp. 112–115] and [32, § 14], makes use of the multiplicative group $H \setminus \{0\}$.

Definition 3.2. Given a pair (M, N) of lines of the projective space $\mathbb{P}(H)$ we say that M is *left parallel* to N , in symbols $M \parallel_L N$, if there is an element $c \in H \setminus \{0\}$ such that $cM = N$. Similarly, M is said to be *right parallel* to N , in symbols $M \parallel_R N$, if $Mc = N$ for some $c \in H \setminus \{0\}$.

The relations \parallel_L and \parallel_R are parallelisms, which are identical precisely in case (B) [32, § 14]. The left and right parallel class of any line M of $\mathbb{P}(H)$ will be written as $\mathcal{S}_L(M)$ and $\mathcal{S}_R(M)$, respectively.

Definition 3.3. The parallelisms \parallel_L and \parallel_R are called the *canonical Clifford parallelisms* of $\mathbb{P}(H)$.

Finally, we extend the previous definition to a projective space $\mathbb{P}(V)$ as in Section 2:

Definition 3.4. A parallelism \parallel of a three-dimensional projective space $\mathbb{P}(V)$ is said to be *Clifford* if the F -vector space V can be made into an F -algebra $H := V$ subject to (A) or (B) such that the given parallelism \parallel coincides with one of the canonical Clifford parallelisms of $\mathbb{P}(H)$.

4 External planes to the Klein quadric

In this section we adopt the settings from Section 2. Our starting point is a very simple one, namely that of a plane C in $\mathbb{P}(V \wedge V)$ external to the Klein quadric. In other words, C has to satisfy the following property:

$$C \text{ has no point in common with the Klein quadric } \mathcal{Q}. \quad (4.1)$$

The restriction to C of the quadratic form ω from (2.3) defines a quadric without points in the projective plane $\mathbb{P}(C)$. Consequently, a plane of this kind cannot exist over certain fields, like quadratically closed fields or finite fields; see [28, p. 4]. The following simple lemma will be used repeatedly.

Lemma 4.1. *Let G be a plane that lies entirely on the Klein quadric \mathcal{Q} , and let $T \supset C$ be a subspace of $\mathbb{P}(V \wedge V)$ with projective dimension k . Then $G \cap T$ has projective dimension $k - 3$.*

Proof. From (4.1) we obtain $G \cap C = 0$, whence $V \wedge V = G \oplus C$ by the dimension formula. Consequently, $V \wedge V = G + T$ and applying again the dimension formula proves the assertion. \square

We now use the given plane C and the Plücker embedding γ from (2.2) to define a parallelism of the projective space $\mathbb{P}(V)$.

Definition 4.2. Given any pair (M, N) of lines in the projective space $\mathbb{P}(V)$ we say that M is *C-parallel* to N , in symbols $M \parallel_C N$, if $C + M^\gamma = C + N^\gamma$. In addition, we define $\mathcal{S}_C(M) := \{X \in \mathcal{L}(V) \mid M \parallel_C X\}$.

Proposition 4.3. *The relation \parallel_C is a parallelism of $\mathbb{P}(V)$. All its parallel classes are regular spreads.*

Proof. Obviously, \parallel_C is an equivalence relation on $\mathcal{L}(V)$. Given any line M and any point p in $\mathbb{P}(V)$ we consider the star $\mathcal{L}(p, V)$. The image $\mathcal{L}(p, V)^\gamma$ is the point set of a plane, say G , lying entirely on \mathcal{Q} . By Lemma 4.1, $G \cap (C + M^\gamma)$ is a single point, whose preimage under γ is the only line through p that is C -parallel to M .

Consider the parallel class $\mathcal{S}_C(M)$ of any line $M \in \mathcal{L}(V)$. Then $\mathcal{S}_C(M)$ is a spread of $\mathbb{P}(V)$. By definition, the image $\mathcal{S}_C(M)^\gamma$ is that quadric in the solid $C + M^\gamma$ which arises as section of the Klein quadric by $C + M^\gamma$. Since $\mathcal{S}_C(M)$ is a

spread, the quadric $\mathcal{S}_C(M)^\gamma$ contains more than one point. From (4.1) there cannot be a line on $\mathcal{S}_C(M)^\gamma$. So the quadric $\mathcal{S}_C(M)^\gamma$ is elliptic. This shows that $\mathcal{S}_C(M)$ is a regular spread. \square

See [1, Lemma 9] and [2, Def. 1.10] for a version of Proposition 4.3 in the classical context. Planes that are external to the Klein quadric arise naturally in elliptic line geometry (over the real numbers) [48, pp. 339–342]. Proposition 4.3 appears also in [24, Sect. 3] in the setting of *generalised elliptic spaces*. However, our current approach shows that the elliptic polarity used there is superfluous when exhibiting a Clifford parallelism on its own. Even more, avoiding such an elliptic polarity in $\mathbb{P}(V)$ allows us to treat the subject in full generality, whereas [24] will not tell us anything about the first case in Proposition 4.6 (c) below.

Proposition 4.4. *For each parallel class \mathcal{S}_C of \parallel_C there is a unique solid, say T , in $\mathbb{P}(V \wedge V)$ such that the section of the Klein quadric by T equals \mathcal{S}_C^γ . Furthermore, the plane C is contained in any such T . Conversely, any solid in $\mathbb{P}(V \wedge V)$ that contains the plane C arises in this way from precisely one parallel class of \parallel_C .*

Proof. By choosing some $M \in \mathcal{S}_C$, we obtain from Definition 4.2 that \mathcal{S}_C^γ is the section of the Klein quadric by the solid $T := C + M^\gamma$. Next, we read off from the proof of Proposition 4.3 that \mathcal{S}_C^γ is an elliptic quadric, whence its span is a solid, which clearly coincides with T . So our T is uniquely determined by \mathcal{S}_C and, by its definition, contains the plane C .

If $T \supset C$ is a solid then, upon choosing a plane G on the Klein quadric \mathcal{Q} and by applying Lemma 4.1, we see that there is a line $M \in \mathcal{L}(V)$ with $M^\gamma = T \cap G$. So, by the above, $\mathcal{S}_C(M)^\gamma$ generates the solid T . Since $\mathcal{S}_C(M)^\gamma$ is the section of \mathcal{Q} by the solid T (and not only a subset of this section), no parallel class other than $\mathcal{S}_C(M)$ gives rise to T . \square

Corollary 4.5. *The plane C in $\mathbb{P}(V \wedge V)$ can be uniquely recovered from any two distinct parallel classes of the parallelism \parallel_C of $\mathbb{P}(V)$.*

Proposition 4.6. *Let C^\perp be the polar plane of C with respect to the Klein quadric \mathcal{Q} . Then the following assertions hold:*

- (a) *The plane C^\perp is external to \mathcal{Q} .*
- (b) *If $\text{Char } F \neq 2$ then the planes C and C^\perp have no point in common.*
- (c) *If $\text{Char } F = 2$ then either $C \cap C^\perp$ is a single point or $C = C^\perp$.*

Proof. Ad (a): Let $q \subset C^\perp$ be a point. Then $q^\perp \supset C$ is a hyperplane in $\mathbb{P}(V \wedge V)$. By Lemma 4.1, $G \cap q^\perp$ is a line for all planes G on the Klein quadric \mathcal{Q} . Any

tangent hyperplane of \mathcal{Q} contains a plane that lies entirely in \mathcal{Q} . Thus q^\perp cannot be tangent to \mathcal{Q} . This in turn shows that $q \notin \mathcal{Q}$.

Ad (b): Due to $\text{Char } F \neq 2$, an arbitrary point p of $\mathbb{P}(V \wedge V)$ belongs to \mathcal{Q} if, and only if, $p \subset p^\perp$. From this observation and (4.1), we obtain $p \not\subset p^\perp \supset C^\perp$ for all points $p \subset C$.

Ad (c): Now $\text{Char } F = 2$ forces the bilinear form $\langle \cdot, \cdot \rangle$ from (2.4) to be symplectic, whence \perp is a null polarity. The restriction of $\langle \cdot, \cdot \rangle$ to $C \times C$ is an alternating bilinear form with radical $C \cap C^\perp$. So the vector dimension of the quotient vector space $C/(C \cap C^\perp)$ has to be even. This implies that $C \cap C^\perp$ is either a point or a plane. In the latter case we clearly have $C = C^\perp$. \square

By Propositions 4.3 and 4.6, the plane C^\perp also gives rise to a parallelism, which will be denoted by \parallel_{C^\perp} . Clearly, the role of C and C^\perp is interchangeable.

Proposition 4.7. *Let $\mathcal{R} \subset \mathcal{L}(V)$ be a regulus whose lines are mutually C -parallel. Then the lines of its opposite regulus \mathcal{R}' are mutually C^\perp -parallel.*

Proof. The image \mathcal{R}^γ is the section of \mathcal{Q} by a (uniquely determined) plane, say E . Then \mathcal{R}'^γ is the section of \mathcal{Q} by the plane E^\perp . Let \mathcal{S}_C denote the C -parallel class that contains \mathcal{R} . By Proposition 4.4, its image \mathcal{S}_C^γ is the section of \mathcal{Q} by a uniquely determined solid, say T , with $T \supset C$. Now $T = E + C$ implies that $E \cap C$ is a line, whence $E^\perp + C^\perp$ is a solid. Thus the lines of \mathcal{R}' are mutually C^\perp -parallel. \square

We are now in a position to show our first main result, namely that $\mathbb{P}(V)$ together with our parallelisms \parallel_C and \parallel_{C^\perp} is a *double space* [30, p. 113], [32, p. 75]. This amounts to verifying the *double space axiom*, which in our setting reads as follows:

- (D) For any three non-collinear points p, q, r in $\mathbb{P}(V)$ the unique line through r that is C -parallel to $p \oplus q$ has a point in common with the unique line through q that is C^\perp -parallel to $p \oplus r$.

Theorem 4.8. *The parallelisms \parallel_C and \parallel_{C^\perp} turn the projective space $\mathbb{P}(V)$ into a double space. This implies that the F -vector space V can be made into an F -algebra $H := V$ subject to (A) or (B) such that the canonical Clifford parallelisms \parallel_L and \parallel_R of $\mathbb{P}(H)$ coincide with \parallel_C and \parallel_{C^\perp} , respectively.*

Proof. With the notation from (D), let $M := p \oplus q$ and $N := p \oplus r$. The set of all lines from $\mathcal{S}_C(M)$ that meet N in some point is a regulus \mathcal{R} , say. One line of \mathcal{R} is the unique line M_1 satisfying $r \subset M_1 \parallel_C M$. The point q is incident with a unique line N_1 of the regulus \mathcal{R}' opposite to \mathcal{R} . Hence M_1 and N_1 have a point in common. From $N \in \mathcal{R}'$ and Proposition 4.7, this N_1 is at the same time the only line through q that satisfies $N_1 \parallel_{C^\perp} N$.

By [32, (14.2) and (14.4)] or [30, Thms. 1–4], where the work of numerous authors is put together, the vector space V can be endowed with a multiplication that makes it into an F -algebra with the required properties. \square

5 From Clifford towards Klein

Let us turn to the problem of reversing Theorem 4.8. We assume that an F -algebra H is given according to condition (A) or (B) from Section 3. We aim at describing the canonical Clifford parallelisms of $\mathbb{P}(H)$ in terms of the ambient space $\mathbb{P}(H \wedge H)$ of the Klein quadric \mathcal{Q} . (Notations that were introduced for $V \wedge V$ will be used *mutatis mutandis* also for $H \wedge H$.) To this end let H/F be the *quotient vector space*³ of the F -vector space H modulo its subspace F . The mapping

$$\beta: H \times H \rightarrow H/F: (g, h) \mapsto \bar{g}h + F \quad (5.1)$$

is F -bilinear and alternating, since for all $h \in H$ the norm $\bar{h}h$ is in $0 + F \in H/F$. By the universal property of the exterior square $H \wedge H$, there is a unique F -linear mapping

$$\kappa: H \wedge H \rightarrow H/F \text{ such that } (g \wedge h)^\kappa = (g, h)^\beta \text{ for all } g, h \in H, \quad (5.2)$$

and we define

$$C := \ker \kappa. \quad (5.3)$$

Our κ is surjective, due to $h + F = (1 \wedge h)^\kappa$ for all $h \in H$. So the kernel of κ has vector dimension $6 - 3$, i. e., $C = \ker \kappa$ is a plane in $\mathbb{P}(H \wedge H)$. In analogy to (5.1) and (5.2), the alternating F -bilinear mapping

$$\beta': H \times H \rightarrow H/F: (g, h) \mapsto g\bar{h} + F \quad (5.4)$$

gives rise to a uniquely determined surjective F -linear mapping $\kappa': H \wedge H \rightarrow H/F$ such that $(g \wedge h)^{\kappa'} = (g, h)^{\beta'}$ for all $g, h \in H$. Therefore, a second plane in $\mathbb{P}(H \wedge H)$ is given by

$$C' := \ker \kappa'. \quad (5.5)$$

Theorem 5.1. *In $\mathbb{P}(H \wedge H)$ the plane $C = \ker \kappa$ is external to the Klein quadric \mathcal{Q} . The canonical Clifford parallelism \parallel_L of $\mathbb{P}(H)$ coincides with the parallelism \parallel_C that arises from the plane C according to Definition 4.2. A similar result holds for the plane $C' = \ker \kappa'$, the canonical Clifford parallelism \parallel_R , and the parallelism $\parallel_{C'}$. Furthermore, C and C' are mutually polar under the polarity of the Klein quadric.*

³The symbol H/F will exclusively be used to denote this quotient space rather than to express that H is a skew field extension of F .

Proof. We choose any point of \mathcal{Q} ; it can be written in the form $F(g \wedge h)$ for some elements $g, h \in H$ that are linearly independent over F . We have

$$g \wedge x \in C \Leftrightarrow \bar{g}x \in F \Leftrightarrow g^{-1}x \in F \Leftrightarrow x \in gF = Fg \text{ for all } x \in H.$$

So the arbitrarily chosen point $F(g \wedge h)$ is not incident with C .

The multiplicative group $H \setminus \{0\}$ acts on H via left multiplication. More precisely, for any $c \in H \setminus \{0\}$ we obtain the *left translation* $\lambda_c: H \rightarrow H: h \mapsto ch$. As λ_c is an F -linear bijection, so is its exterior square $\lambda_c \underline{\wedge} \lambda_c: H \wedge H \rightarrow H \wedge H$. This exterior square describes the action of λ_c on the line set $\mathcal{L}(H)$ in terms of bivectors, as it takes any pure bivector $g \wedge h \in H \wedge H$ to $cg \wedge ch$. So we may read off from $\bar{c}c = c\bar{c} \in F \setminus \{0\}$ and

$$(cg \wedge ch)^\kappa = \bar{c}gch + F = \bar{g}(\bar{c}c)h + F = (c\bar{c})(g \wedge h)^\kappa$$

that

$$C + F(cg \wedge ch) = C + F(g \wedge h) \text{ for all } g, h \in H. \quad (5.6)$$

As the pure bivectors span $H \wedge H$, formula (5.6) implies that all subspaces of $H \wedge H$ passing through C are invariant under $\lambda_c \underline{\wedge} \lambda_c$.

Now let $M \parallel_L N$, whence there is a particular $c \in H \setminus \{0\}$ with $cM = N$. By the above, the subspace $C + M^\gamma$ is invariant under $\lambda_c \underline{\wedge} \lambda_c$, so that $C + M^\gamma = C + (cM)^\gamma$ or, said differently, $M \parallel_C N$. We obtain as an intermediate result that every left parallel class is a subset of a C -parallel class. However, a left parallel class cannot be properly contained in a C -parallel class, for then it would not cover the entire point set of $\mathbb{P}(H)$.

By switching from left to right and replacing κ with κ' , the above reasoning shows that the right parallelism \parallel_R coincides with $\parallel_{C'}$.

Finally, we establish that the polarity \perp of the Klein quadric takes the plane C to the plane C' . Here we use the well known result that each of the two (not necessarily distinct) parallelisms of a projective double space determines uniquely the other parallelism [32, p. 76]. So, by the above and Theorem 4.8, we obtain from $\parallel_L = \parallel_C$ that $\parallel_{C'} = \parallel_R = \parallel_{C^\perp}$. Now Corollary 4.5 shows $C' = C^\perp$. \square

By virtue of Theorem 4.8 and the preceding theorem, we have established the announced one-one correspondence between Clifford parallelisms and planes that are external to the Klein quadric.

Remark 5.2. As our approach to the planes C and $C' = C^\perp$ in (5.3) and (5.5) is somewhat implicit, it seems worthwhile to write down a basis for each of these planes. This can be done as follows: First we apply κ and κ' to the six basis elements $1 \wedge i, 1 \wedge j, \dots, j \wedge k$ of $H \wedge H$, which then allows us to find three linearly independent bivectors in $\ker \kappa$ and $\ker \kappa'$, respectively. See Table 1 and take notice that in case (A2) the point $C \cap C^\perp$ is given by the bivector $b \wedge i + j \wedge k$.

Cases	Plane	Basis		
(A1), (B)	C	$b \wedge i - j \wedge k$	$a \wedge j + i \wedge k$	$1 \wedge k + i \wedge j$
	C^\perp	$b \wedge i + j \wedge k$	$a \wedge j - i \wedge k$	$1 \wedge k - i \wedge j$
(A2)	C	$b \wedge i + j \wedge k$	$a \wedge j + i \wedge k$	$1 \wedge (k + j) + i \wedge j$
	C^\perp	$b \wedge i + j \wedge k$	$1 \wedge (aj + k) + i \wedge k$	$1 \wedge k + i \wedge j$

Table 1: Bases of the planes C and C^\perp .

Alternatively, we may consider hyperplanes of $\mathbb{P}(H \wedge H)$ that are incident with C . Any such hyperplane can be obtained as the kernel of an F -linear form $H \wedge H \rightarrow F$ as follows: We choose any non-zero F -linear form $\varphi: H \rightarrow F$ such that $1^\varphi = 0$. Then the mapping

$$H \times H \rightarrow F: (g, h) \mapsto (\bar{g}h)^\varphi \quad (5.7)$$

is an alternating F -bilinear form. The universal property of $H \wedge H$ gives the existence of a unique F -linear form $\psi: H \wedge H \rightarrow F$ such that $(g \wedge h)^\psi = (\bar{g}h)^\varphi$ for all $g, h \in H$, and clearly $C \subset \ker \psi$. In this way C can be described as intersection of three appropriate hyperplanes. By replacing $\bar{g}h$ with $g\bar{h}$, a similar result is obtained for C^\perp . In case (A2), *i. e.*, if H is a quaternion skew field and $\text{Char } F = 2$, the trace form is a distinguished choice of φ . It turns (5.7) into the polar form of the norm: $(g, h) \mapsto \bar{g}h + \bar{h}g$. Furthermore, the hyperplane arising from the trace form is equal to $C + C^\perp$, due to $\bar{g}h + \bar{h}g = g\bar{h} + h\bar{g}$ for all $g, h \in H$.

Remark 5.3. The mappings (5.1) and (5.4) admit a geometric interpretation by considering the projective plane $\mathbb{P}(H/F)$. The “points” of this projective plane can be identified with the lines of the star $\mathcal{L}(F1, H)$ via $F(h + F) \mapsto F1 \oplus Fh$ for all $h \in H \setminus F$. We define a mapping $\mathcal{L}(H) \rightarrow \mathcal{L}(F1, H)$ by assigning to each line M the only line of the parallel class $\mathcal{S}_L(M)$ through the point $F1$. Letting $M = Fg \oplus Fh$ with (linearly independent) $g, h \in H$, we obtain $F1 \oplus F(g^{-1}h) = F(\bar{g}h) \oplus F1$ as image of M . So the vector $\bar{g}h + F \in H/F$ appearing in (5.1) is a representative of the image of M . The interpretation of (5.4) in terms of $\|\cdot\|_R$ is similar. In the classical setting such a mapping is known under its German name *Eckhart-Rehbock Abbildung*; see [9], [49], and the references given there. Generalisations, in particular to Lie groups, are the topic of [9, pp. 16–17], [29], and [35].

Alternating mappings like the ones from (5.1) and (5.4) appear in the definition of *generalised Heisenberg algebras*; see, for example, [45, Def. 6.1]. These algebras are important in the classification of certain *nilpotent Lie algebras*. We

encourage the reader to take a closer look at [10], [19, Sect. 7], and [45, Sect. 8], in order to see how planes external to the Klein quadric have successfully been utilised in that context.

Remark 5.4. Our proof of Theorem 5.1 has shown the following: All hyperplanes of $\mathbb{P}(H \wedge H)$ that are incident with C are invariant under the exterior square of any left translation λ_c . Since $\lambda_c \triangle \lambda_c$ commutes with the polarity of the Klein quadric, we immediately obtain that $\lambda_c \triangle \lambda_c$ fixes all points of the plane C^\perp . A similar result holds for the exterior square of any *right translation* $\rho_c: H \rightarrow H: h \mapsto hc$. However, we do not enter into a detailed discussion of these mappings. In this regard, the articles [18] and [50] about real quaternions deserve special mention.

6 Characterisations of a single Clifford parallelism

In this section we consider again a three-dimensional projective space $\mathbb{P}(V)$ as described in Section 2.

Lemma 6.1. *No spread of $\mathbb{P}(V)$ is contained in a special linear complex of lines.*

Proof. Assume to the contrary that a spread \mathcal{S} is contained in a special linear complex with axis $A \in \mathcal{L}(V)$, say. Choose any point p_1 off the line A and let $M_1 \in \mathcal{S}$ be the line through p_1 . By our assumption, $A + M_1$ is a plane, and in this plane there is a point p_2 that lies neither on A nor on M_1 . The plane $A + M_1$ contains also the line $M_2 \in \mathcal{S} \setminus \{M_1\}$ through p_2 , whence M_1 and M_2 have a unique common point, an absurdity. \square

We add in passing that Lemma 6.1 is closely related with a result [15, Prop. 6.10 (4)] about a specific class of geometric hyperplanes arising from regular spreads of lines (for arbitrary odd projective dimension). We now show our first characterisation of a Clifford parallelism:

Theorem 6.2. *For any parallelism \parallel of $\mathbb{P}(V)$ the following properties are equivalent:*

- (a) *The parallelism is Clifford.*
- (b) *Any two distinct parallel classes are contained in at least one geometric hyperplane of the Grassmann space $\mathbb{G}_1(V)$.*
- (c) *Any two distinct parallel classes are contained in a unique linear complex of $\mathbb{P}(V)$. This linear complex is general.*

Proof. (a) \Rightarrow (b): By Definition 3.4, we can apply Theorem 5.1. This shows that \parallel is one of the parallelisms \parallel_C and $\parallel_{C'}$ from there. So, up to a change of notation, we may assume $\parallel = \parallel_C$. Suppose that $\mathcal{S}_1 \neq \mathcal{S}_2$ are parallel classes of \parallel . The image \mathcal{S}_1^γ is the section of the Klein quadric by a solid $T_1 \supset C$ according to Definition 4.2. Likewise, all points of \mathcal{S}_2^γ are contained in a solid $T_2 \supset C$. Since T_1 and T_2 have the plane C in common, there exists a hyperplane W of $\mathbb{P}(V \wedge V)$ such that $W \supset T_1 + T_2$. This W is incident with all points of $\mathcal{S}_1^\gamma \cup \mathcal{S}_2^\gamma$, and so $\{X \in \mathcal{L}(V) \mid X^\gamma \subset W\}$ is a linear complex of lines (or, said differently, a geometric hyperplane) containing $\mathcal{S}_1 \cup \mathcal{S}_2$.

(b) \Rightarrow (c): Suppose that parallel classes $\mathcal{S}_1 \neq \mathcal{S}_2$ are both contained in at least one geometric hyperplane. Let \mathcal{H} be any of these. We have noticed in Section 2 that \mathcal{H} is a linear complex of lines which, by Lemma 6.1 applied to \mathcal{S}_1 , has to be a general.

Now let \mathcal{H}' and \mathcal{H}'' be general linear complexes of lines both containing $\mathcal{S}_1 \cup \mathcal{S}_2$. Choose any point p of $\mathbb{P}(V)$. Through p there are uniquely determined lines $M_1 \in \mathcal{S}_1$ and $M_2 \in \mathcal{S}_2$. Also we have $M_1 \neq M_2$. Each of the intersections $\mathcal{H}' \cap \mathcal{L}(p, V)$ and $\mathcal{H}'' \cap \mathcal{L}(p, V)$ is a pencil of lines. Both pencils have to contain the lines M_1 and M_2 , and therefore these pencils are identical. As p varies in the point set of $\mathbb{P}(V)$, this gives $\mathcal{H}' = \mathcal{H}''$.

(c) \Rightarrow (a): Choose any parallel class $\mathcal{S}(M)$ with $M \in \mathcal{L}(V)$. Our first aim is to show that the points of $\mathcal{S}(M)^\gamma$ generate a solid in $\mathbb{P}(V \wedge V)$. Let q be a point of M . Through q there are lines M_1 and M_2 such that M , M_1 , and M_2 are not coplanar. Denote by $\mathcal{H}(M, M_1)$ and $\mathcal{H}(M, M_2)$ the uniquely determined general linear complexes that contain $\mathcal{S}(M) \cup \mathcal{S}(M_1)$ and $\mathcal{S}(M) \cup \mathcal{S}(M_2)$, respectively. Also let W_1 and W_2 be the hyperplanes of $\mathbb{P}(V \wedge V)$ corresponding to these linear complexes. Then $\mathcal{H}(M, M_1) \cap \mathcal{L}(q, V)$ is a pencil of lines, which contains M and M_1 , but not M_2 . Thus $\mathcal{H}(M, M_1) \neq \mathcal{H}(M, M_2)$. This gives $W_1 \neq W_2$, and therefore $T := W_1 \cap W_2$ turns out to be a solid of $\mathbb{P}(V \wedge V)$. The γ -preimage of the point set $\mathcal{P}(T)$ is a linear congruence of lines of $\mathbb{P}(V)$, say \mathcal{E} , which contains the spread $\mathcal{S}(M)$ as a subset. No hyperplane of $\mathbb{P}(V \wedge V)$ through T can be tangent to the Klein quadric by Lemma 6.1, whence the line T^\perp is exterior to the Klein quadric. This means that the linear congruence \mathcal{E} is elliptic and hence a regular spread. As the spread $\mathcal{S}(M)$ is contained in the spread \mathcal{E} , these two spreads are identical. Therefore $\mathcal{S}(M)^\gamma$, due to its being an elliptic quadric, generates the solid T .

According to the previous paragraph, we may assign to each parallel class \mathcal{S} a uniquely determined solid T of $\mathbb{P}(V \wedge V)$. Let \mathcal{F} be the set of all such solids. The assignment $\mathcal{S} \mapsto T$ is injective, since \mathcal{S}^γ is the section of the Klein quadric by T . Hence \mathcal{F} comprises more than one solid. From our assumption in (c), any two distinct solids from \mathcal{F} are contained in a unique hyperplane of $\mathbb{P}(V \wedge V)$. This implies (see, for example, [38, Prop. 3.2]) that at least one of the following assertions holds:

- (i) There is a unique hyperplane of $\mathbb{P}(V \wedge V)$ that is incident with all solids belonging to \mathcal{F} .
- (ii) There is a unique plane of $\mathbb{P}(V \wedge V)$ that is incident with all solids belonging to \mathcal{F} .

The situation from (i) cannot occur in our setting, since it would imply that all parallel classes, and hence all of $\mathcal{L}(V)$, would belong to a single linear complex of lines. Consequently, only (ii) applies, and we let C be this uniquely determined plane. We establish that this C is external to the Klein quadric: Indeed, any common point would be the γ -image of a line belonging to *all* parallel classes, which is utterly absurd. Consequently, \parallel coincides with \parallel_C , and Theorem 4.8 shows that \parallel is Clifford. \square

For a version of the previous result, limited to the real case and with somewhat different assumptions, we refer to [1, Lemma 14] and [2, Def. 1.9]. Our second characterisation is based on the following *condition of crossed pencils* (Figure 1) for an arbitrary parallelism \parallel of $\mathbb{P}(V)$:

- (CP) For all lines $M_1 \parallel N_1$ and $M_2 \parallel N_2$ such that $p := M_1 \cap M_2$ and $q := N_1 \cap N_2$ are two distinct points the following holds: If the lines M_1 , M_2 , and $p \oplus q$ are in a common pencil, so are the lines N_1 , N_2 , and $p \oplus q$.

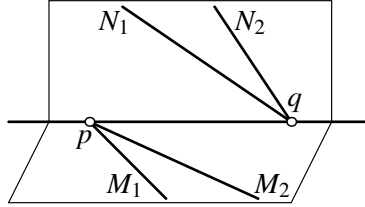


Figure 1: Condition of crossed pencils

Theorem 6.3. *A parallelism \parallel of $\mathbb{P}(V)$ is Clifford if, and only if, it satisfies the condition of crossed pencils.*

Proof. Let \parallel be Clifford. We consider lines M_1, M_2, N_1, N_2 subject to the assumptions in (CP). By Theorem 6.2, there is a unique general linear complex of lines containing $\mathcal{S}(M_1) \cup \mathcal{S}(M_2)$. This complex is the set of null lines of a null polarity π , say. So $p^\pi = (M_1 \cap M_2)^\pi = M_1 + M_2$ and $q^\pi = (N_1 \cap N_2)^\pi = N_1 + N_2$. Now if $M_1, M_2, p \oplus q$ are in a common pencil of lines then $q \subset p^\pi$ implies $p \subset q^\pi$, and this shows that N_1, N_2 , and $p \oplus q$ belong to the pencil $\mathcal{L}(q, q^\pi)$.

For a proof of the converse we consider any two distinct parallel classes \mathcal{S}_1 and \mathcal{S}_2 . For each point p in $\mathbb{P}(V)$ we define $M_1(p)$ as the only line satisfying $p \subset M_1(p) \in \mathcal{S}_1$; the line $M_2(p)$ is defined analogously. We obtain a well defined mapping π of the point set $\mathcal{P}(V)$ into the set of planes of $\mathbb{P}(V)$ via

$$p \mapsto p^\pi := M_1(p) + M_2(p).$$

We claim that $q \subset p^\pi$ implies $p \subset q^\pi$ for all $p, q \in \mathcal{P}(V)$: This is trivially true when $p = q$ and immediate from (CP) otherwise. Consequently, π is a polarity of $\mathbb{P}(V)$, which is null by its definition. The set of null lines of π is a general linear complex of lines (a geometric hyperplane of $\mathbb{G}_1(V)$) containing $\mathcal{S}_1 \cup \mathcal{S}_2$. By virtue of Theorem 6.2, the parallelism \parallel is Clifford. \square

Remark 6.4. The condition of crossed pencils can readily be translated to an *arbitrary* three-dimensional projective space $(\mathcal{P}, \mathcal{L})$, and it could be used to define when a parallelism of this space is *Clifford*. As in the proof of Theorem 6.3, the existence of such a Clifford parallelism implies that the projective space $(\mathcal{P}, \mathcal{L})$ admits a null polarity, which in turn forces $(\mathcal{P}, \mathcal{L})$ to be Pappian.

7 Conclusion

By our investigation, which is far from being comprehensive, there is a one-one correspondence between Clifford parallelisms and external planes to the Klein quadric. We have not included several topics. Among these is the problem of finding necessary and sufficient algebraic conditions for two parallel classes of a given Clifford parallelism to be projectively equivalent, even though all necessary tools can be found in the literature: In the setting from Section 5 there are one-one correspondences among (i) quadratic extension fields of F that are contained in H (which are precisely the lines of $\mathbb{P}(V)$ through the point $F1$), (ii) parallel classes of $\parallel_L = \parallel_C$, (iii) solids through C , and (via \perp) (iv) lines in the plane C^\perp . More generally, any external line to the Klein quadric can be linked directly with a quadratic extension field of F and vice versa. We refer to [3], [10], [14], [15], [19], [22], [23], and [45] for a wealth of (overlapping) results that should settle the issue. Also we have not incorporated the results from [8] and [25], where Clifford parallelisms have been described by extending the ground field. Finally, we are of the opinion that *kinematic line mappings* and related work from [9], [21], [35], and [37] could provide a good guideline for a generalisation of our findings to other parallelisms arising from *kinematic spaces*; see [32], [39], and the references therein.

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